Transient Solution of Time-Variant Structural Systems Using Invariant Modal Properties

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This paper presents a general methodology for the solution of the transient responses for structural systems with time-varying properties, i.e., system modal properties changing with time. The method presented solves the modal equations of motion while treating the time-variant system properties as applied forces. The advantage of this solution technique is that only one set of system modal properties needs to be calculated. The particular modal equation of motion solution technique used in this paper assumes that the forces imposed by the varying structural properties are a linear function of time between any two solution time steps. The accuracy is thus dependent on the time step size and convergence is easily tested. Three sample problems are presented that demonstrate that this approach is easily accomplished for at least certain types of time-variant structural problems. Modal truncation effects are discussed, and it is demonstrated that, with corrections for truncated flexibility, time-variant modal systems can be solved accurately by using one set of invariant modal properties.

Nomenclature

A,A',B,B'	= coefficient of integration
D	= all known variables to the modal displacement solution
dt	= time step size
F,F'	= coefficient of integration
f	= physical force induced by the time-varying structural parameters
f	= externally applied physical force
G,G'	= coefficient of integration
I	= identity matrix
K	= discrete physical stiffness matrix
L	= finite element beam length
M	= physical mass
m	= physical mass distribution
P	= generalized force
q	= beam element physical nodal displacements
t	= time
V	= all known variables to the modal velocity solution
w	= transverse displacement
x	= coordinate direction
β	= modal damping coefficient
Δ	= displacement or time difference
ξ	= modal displacement degree of freedom
ϕ	= eigenvectors
ω	= eigenvalues
0	= null matrix

Subscripts

b	= beam parameter
i,k	= modal degree of freedom
j	= physical degree of freedom
n	= time step
o	= eigenvalue/generalized mass
p	= pallet parameter

Superscripts

i	= inertial force
s	= spring force
T	= matrix transpose

Introduction

THE transient solutions for time-invariant structural problems are usually performed in modal coordinates that are obtained from an eigensolution. The primary advantages include uncoupling the equations of motion, resulting in a direct and efficient solution, and reducing the problem size by enabling the analyst to discard the mode shapes that are negligent to the solution, usually the higher order modes. The fact that the eigenvalues correlate to modal frequencies and eigenvectors to mode shapes makes possible the structural testing of math model accuracy. An accurate math model is certainly a necessary condition to obtain accurate analytical results. Thus, it is necessary that in the solution of time-variant structural transients the analyst account for these changes in the system's modal properties. However, it may not be necessary to perform multiple eigensolutions.

Some interest in time-variant structural dynamic solutions was generated by the Space Transportation System (STS) payload community. The use of sliding trunnions to support the payload systems to the Space Shuttle allows for friction-induced loads, with its resulting sticking and sliding. A trunnion in the static friction state (i.e., stuck or locked) constitutes a load path that alters the STS/payload nonfriction system modal properties. Such systems are very large, usually ~ 1000 degrees of freedom (DOF), and it was not plausible to recalculate system modes for each change in the friction state. Any practical solution technique would need to use an invariant set of eigenvalues and eigenvectors while at the same time accurately account for the friction-induced changes in the structural system's modal characteristics. Reference 2 presented such a methodology.

Reference 3 addressed the dynamic lift-off problem of a booster/spacecraft system separating from its launch pad. This type of problem is also time variant in that the constraints between the booster and pad will vary as the booster's engines reach full thrust and lift the system away from the pad. The approach is much the same as that presented in Ref. 2, but instead of using a finite difference integrator, the method of Ref. 3 uses the modal solution technique most often used for linear time-invariant problems. This leads to the solution of a

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set of coupled equations, but the size of the necessary solution is kept small by use of static condensation.

As was pointed out in Ref. 4, the Space Station program has generated considerable interest in the solution of time-variant problems. Two sample problems were presented in Ref. 4 and solved by several methods using time-dependent modal properties. Not only were time-dependent eigensolutions solved for, but the first and second time derivatives of the eigenvectors were calculated by use of a cubic spline interpolation. Such an approach to the transient friction solution of Ref. 2 is impossible since the transition in modal states from sliding friction to static friction happens instantaneously.

The paper presents solutions to the two problems very similar to those of Ref. 4 and avoids the complications of using time-dependent modal properties. The methods used are similar to those presented in Ref. 3. In addition, problem 2 is expanded into a more complicated system and is also solved. Results are presented that demonstrate the solution convergence with the time step size. Modal truncation effects are then discussed. It is demonstrated that with the use of residual modes, which correct for truncated flexibility, time-variant modal systems can be solved accurately by using a set of invariant modal properties.

Transient Solution

The modal equations of motion can be solved by numerous solution techniques. A very large number of finite difference approaches exist. However, for the case of linear time-invariant structural modal equations of motion, a more computationally efficient method is to use a closed-form solution. This is only possible because the modal equations of motion are uncoupled, i.e., each mode acts as an independent single degree-of-freedom oscillator. Such an approach requires that the time history of the externally applied forces be defined as a function of time. This is seldom the case. Most often, the forces are defined for specific time slices. The solution technique most often used (at least by NASTRAN users) assumes that the applied forces vary linearly between solution time steps. With this one assumption, the solution is derived in a closed-form method. This incremental solution is given by the following three equations:

$$\xi_{i,n+1} = F\,\xi_{i,n} + G\dot{\xi}_{i,n} + AP_{i,n} + BP_{i,n+1} \tag{1}$$

$$\dot{\xi}_{i,n+1} = F'\xi_{i,n} + G'\dot{\xi}_{i,n} + A'P_{i,n} + B'P_{i,n+1} \tag{2}$$

$$\ddot{\xi}_{i,n+1} = P_{i,n+1} - 2\beta\dot{\xi}_{i,n+1} - \omega_o^2\xi_{i,n+1} \tag{3}$$

Equation (1) is the solution for the modal displacement, Eq. (2) for the modal velocity, and Eq. (3) for the modal acceleration. The derivation of these equations and the coefficients of integration for underdamped, critically damped, overdamped, and undamped rigid-body modes can be found in Ref. 1. The simplicity of this solution is derived from the fact that the response for any one mode is uncoupled from all the rest. Again, with the one assumption that the force input varies linearly between solution time steps, the solution is mathematically exact. Although the following derivation is based on Eqs. (1-3), it could proceed from virtually any solution technique.

In normal linear analyses, the generalized force input P is known. The right sides of the equations are defined, making the solution for the $n+1$ time step a trivial task. The solution technique reported in this paper will treat the time-variant parameters as forces acting on the structure. In this manner, and with some care, the changes in vibrational frequencies and mode shapes can be accurately accounted for while still using one eigensolution. In general, these forces and the response are interrelated (i.e., they are dependent upon each other) and they will both remain unknown until the solution for the $n+1$ time step is completed. Equations (1-3) are rewritten with these forces remaining as variables:

$$\xi_{i,n+1} = F\xi_{i,n} + G\dot{\xi}_{i,n} + AP_{i,n} + BP_{i,n+1} + A\phi_{i,j}^T f_{j,n} + B\phi_{i,j}^T f_{j,n+1} \tag{4}$$

$$\dot{\xi}_{i,n+1} = F'\xi_{i,n} + G'\dot{\xi}_{i,n} + A'P_{i,n} + B'P_{i,n+1} + A'\phi_{i,j}^T f_{j,n} + B'\phi_{i,j}^T f_{j,n+1} \tag{5}$$

$$\ddot{\xi}_{i,n+1} = P_{i,n+1} - 2\beta\dot{\xi}_{i,n+1} - \omega_o^2\xi_{i,n+1} + \phi_{i,j}^T f_{j,n+1} \tag{6}$$

The variable f is the physical force(s) induced by the time-varying structural parameters, and ϕ is the corresponding eigenvector row partition(s). Note that the products $B\phi^T$ and $B'\phi^T$ are not typical matrix products. The B and B' coefficients of integration are actually scalar multipliers of P, i.e., B_kP_k. Thus, in Eqs. (4-6) and those to follow, $B\phi^T$ denotes multiplication of row k of ϕ^T by the scalar B_k. The sample problems will demonstrate how these unknown forces can be included as part of the solution. First, Eqs. (4-6) are rewritten in shorter form by placing all of the known variables together:

$$\xi_{i,n+1} = D + B\phi_{i,j}^T f_{j,n+1} \tag{7}$$

$$\dot{\xi}_{i,n+1} = V + B'\phi_{i,j}^T f_{j,n+1} \tag{8}$$

$$\ddot{\xi}_{i,n+1} = P_{i,n+1} - 2\beta\dot{\xi}_{i,n+1} - \omega_o^2\xi_{i,n+1} + \phi_{i,j}^T f_{j,n+1} \tag{9}$$

Example Problem 1

Example problem 1, adapted from Ref. 4, consists of a simple 2-DOF system with one time-varying mass and one time-varying spring. The variation of the mass and spring with time are known. The forces imposed on the system by these time variations will be dependent on both the instantaneous displacement and acceleration states. Figure 1 illustrates the problem.

This problem has no external force application (i.e., $P=0$), zero initial velocities, with initial displacements of 10 on both masses. The eigensolution used to uncouple the equations of motion will be based on the time zero system properties. The inertial force acting on DOF 2 to time-dependent mass is given by Eq. (10):

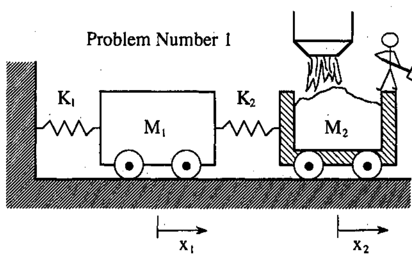
$$M(t)_2 * \ddot{x}_2 = -f_2^i \tag{10}$$

This equation is rewritten into the coordinates from the eigensolution:

$$M(t)_2 * [\phi_2]\{\ddot{\xi}\} = -f_2^i \tag{11}$$

The time-varying contribution from the variable spring is given by Eq. (12):

$$K(t) * (x_2 - x_1) = f^s \tag{12}$$



$K_1 = 10$
$K_2 = 10 - 9 * t/150$
$M_1 = 10$
$M_2 = 51 + 50 * \sin(3 * \pi * t/150)$

Fig. 1 Discrete 2 DOF problem.

Again, this is transformed into the time zero eigensolution coordinate system:

$$K(t) * [\phi_2 - \phi_1]\{\xi\} = f^s \tag{13}$$

Substitution of Eqs. (11) and (13) into Eqs. (7–9), and placing all unknowns on the left side yields the following coupled matrix equation:

$$\begin{bmatrix} I & & & -B\phi_2^T & -B[\phi_1-\phi_2]^T \\ & I & & -B'\phi_2^T & -B'[\phi_1-\phi_2]^T \\ \omega_o^2 & 2\beta & I & -\phi_2^T & -[\phi_1-\phi_2]^T \\ & & M(t)_2*[\phi_2] & I & \\ K(t)*[\phi_1-\phi_2] & & & & I \end{bmatrix} \begin{Bmatrix} \xi \\ \dot{\xi} \\ \ddot{\xi} \\ f_2^i \\ f^s \end{Bmatrix} = \begin{Bmatrix} D \\ V \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{14}$$

With Eq. (14), the problem can be solved. The solution proceeds time step by time step, updating the square matrix each time step. The right side of the equation is updated using the known quantities D and V, P being zero for this problem. Note that this problem is small, only requiring 8 DOF, i.e., two modal displacements, two modal velocities, two modal accelerations, and two forces.

Problem 1 Solution

The time zero eigenvalues and frequencies are listed in Table 1. The problem 1 time history solutions for three different time steps are plotted in Fig. 2. The solutions demonstrate excellent convergence with the time step size. In fact, the solutions for the first mass with time step sizes of 0.5 and 0.1 are almost indistinguishable.

Table1 Problem 1 time zero eigenvalues

Mode	Eigenvalue	Frequency, Hz
1	0.0932	0.0486
2	2.1028	0.2308

Accuracy of System Modal Properties

Solving transient problems by use of modes has the advantages of both uncoupling the equations of motion and, in most cases, allows for problem reduction through modal truncation. As was stated in the Introduction, the solution of time-variant structural transients requires that the analyst account for the changes in the system's modal properties, and yet, in the above solution, only one set of modal properties is used. How were the changes in modal characteristics accurately accounted for? Basically, transient problems are solved by using modes because the modal coordinate system is computationally efficient. The uncoupled nature of the modal degrees of freedom leads to a closed-form solution approach, the only assumption being that all external forces vary linearly between the solution time steps. In the previous solution, there was no modal truncation, i.e., no information from the time zero discrete physical model had been discarded. The time-varying parameters were treated as externally applied forces. This solution showed excellent convergence with time step size. Updating the modal properties would have simply been a mathematical burden resulting from the analyst's choice of continually changing the solution coordinate system.

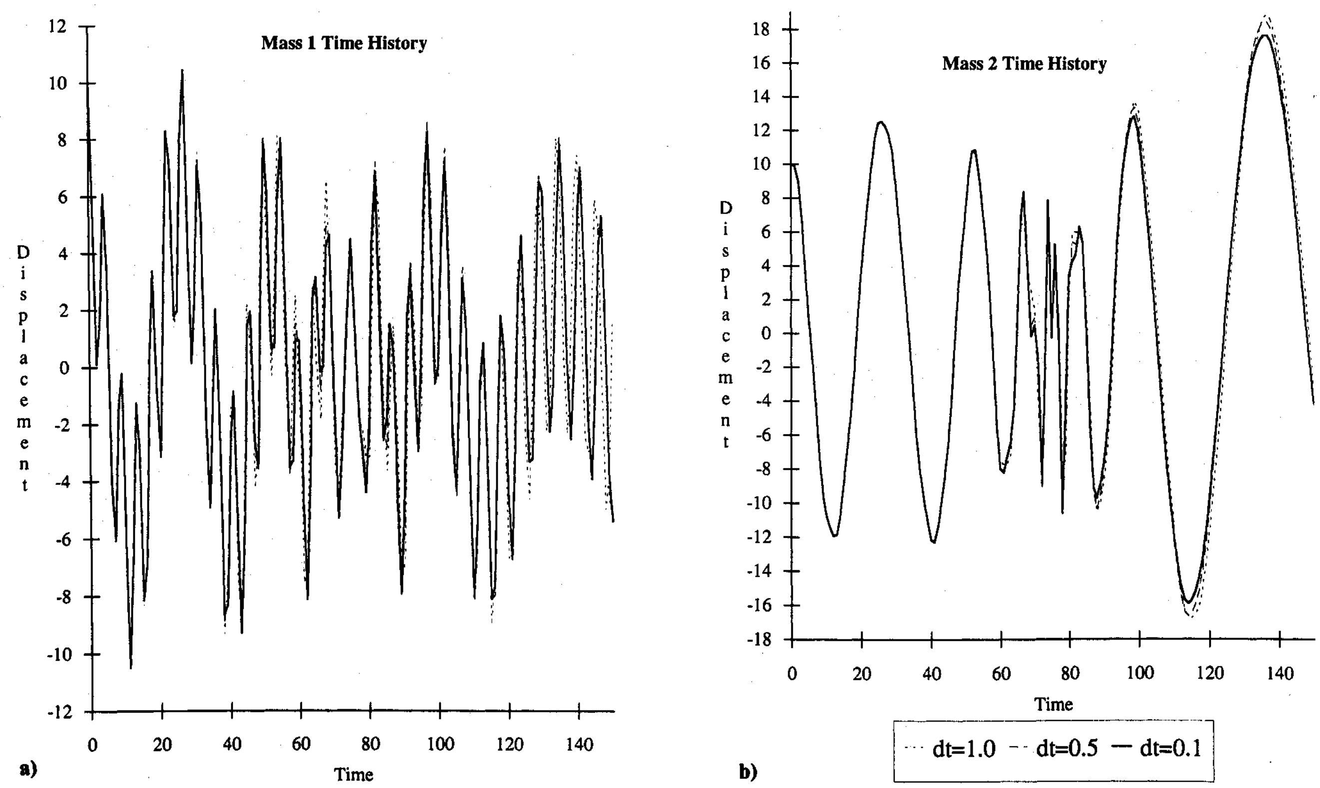


Fig. 2 Problem 1 solution.

Of course, larger problems will no doubt have significant modal truncation. For example, the problem of a moving pallet on a truss structure cantilevered from a Space Station math model with 1000 DOF, or even more, could be truncated down to just a few hundred modes or even less. Solving the moving pallet transients by the methods mentioned earlier would not yield accurate results because the truss structure had been artificially stiffened by modal truncation. The same problem arises when including trunnion friction in the calculation of STS/payload transients and also the booster to pad separation problem. References 2 and 3 avoided this inaccuracy by employing the principles of modal synthesis.

One of the most widely used methods of modal synthesis is the improved component-mode representation given by Rubin.[5] This modal synthesis method accounts for the flexibility and inertia at subsystems' interfaces that was lost through modal truncation. With this, the subsystems can be coupled and an accurate system eigensolution calculated. In the large Space Station problem mentioned earlier the analyst could use Rubin's method of including the residual flexibility and inertia to correct for that which was truncated on the truss structure. This would require additional degrees of freedom. With this, the analyst could then calculate a new set of truss structure/ Space Station modes. These new modes will be identical to the original untruncated set, with the addition of a set of residual modes. With no truncation, this new modal system will have all the necessary properties, on all the necessary truss structure degrees of freedom, to calculate accurate system modes for any pallet location. These residual modes are not real vibrational modes, but are eigenvalues and eigenvectors that correct for modal truncation effects at the desired physical degrees of freedom. Again, accurate system modes, within the limits imposed by the original modal trucation, can then be calculated for any desired pallet location. Reference 5 suggests that the substructures' modal content be accurate up to 1.5 times the highest system frequency of interest.

The solution method of this paper avoids the recalculation of system modal properties by treating the pallet as an external force acting on the truss. However, with the use of residual modes, all the necessary structural information needed to calculate accurate system modal properties is contained within the solution's formulation.

The authors have used this approach to solve large STS/ payload transients including trunnion friction effects.[2] As stated before, the residual modes do not represent real vibration modes. Experience has shown that they are best treated quasistatically (i.e., massless). This is consistent with the fact that not including residual inertias in Rubin's method results in system eigensolutions almost identical to those accounting for the residual inertia. This method of modal synthesis is most often referred to as the Rubin-MacNeal method. Thus, the residual inertia effects are typically of a lesser magnitude than the residual flexibility effects. For these reasons, residual inertia is used to calculate the residual modes, which are then treated quasistatically in the transient solution. In this manner, the solution of dynamic systems with coupling between subsystems with time-varying interfaces will be employing the Rubin-MacNeal method of modal synthesis. Treating modal degrees of freedom quasistatically is accomplished easily by using the following coefficients of integration: $F = G = A = 0$, and $B = 1/\omega^2$, with the modal velocity and acceleration equal to zero by definition.

Example Problem 2

The second example problem, also adapted from Ref. 4 and shown in Fig. 3, represents a cantilevered Space Station truss structure with a moving pallet. The pallet mass moves along the truss structure at a constant rate of 5 ft/s, starting at the tip. The truss is idealized using simple beam bending finite elements with 10 elements and 20 DOF. The pallet mass is idealized as a concentrated or point mass. There is no damping. The solution for this problem will use the truss structure's eigensolution with zero pallet mass. The pallet will be treated as a free-free modal system. Treating the pallet as such results in one rigid-body mode. Displacement compatibility is enforced between the truss structure and the pallet by use of the following equations:

$$w_b = [\phi_b]\{\xi_b\} \tag{15}$$

$$w_p = [\phi_p]\{\xi_p\} \tag{16}$$

$$\Delta w = w_p - w_b \tag{17}$$

$$\Delta w = 0 \tag{18}$$

In Eqs. (15–18) the subscript p denotes the pallet. The subscript b denotes the truss structure or beam and ϕ_b is the truss structure eigenvector for the degree of freedom at the location of the pallet.

The modal solution for the truss and pallet is given by Eqs. (7–9), with the appropriate subscripts added. The constraints defined by Eqs. (15–18) are incorporated into the solution. Note that the interface force f has an opposite sign on the pallet. The resulting coupled system results in Eq. (19):

$$\begin{bmatrix}
\diagdown I & & & & & & & & & -B_b\phi_b^T \\
 & \diagdown I & & & & & & & & -B_b'\phi_b^T \\
\diagdown\omega_{o_b}^2 & \diagdown 2\beta_b & \diagdown I & & & & & & & -\phi_b^T \\
 & & & \diagdown I & & & & & & B_p\phi_p^T \\
 & & & & \diagdown I & & & & & B_p'\phi_p^T \\
 & & & \diagdown\omega_{o_p}^2 & \diagdown 2\beta_p & \diagdown I & & & & \phi_p^T \\
 & & & -\phi_p & & & \diagdown I & & & \\
-\phi_b & & & & & & & \diagdown I & & \\
 & & & & & & \diagdown -I & \diagdown I & \diagdown I & \\
 & & & & & & & & \diagdown I & \diagdown 0
\end{bmatrix}
\begin{Bmatrix} \xi_b \\ \dot{\xi}_b \\ \ddot{\xi}_b \\ \xi_p \\ \dot{\xi}_p \\ \ddot{\xi}_p \\ w_p \\ w_b \\ \Delta w \\ f \end{Bmatrix}
=
\begin{Bmatrix} D_b \\ V_b \\ 0 \\ D_p \\ V_p \\ P_p \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{19}$$

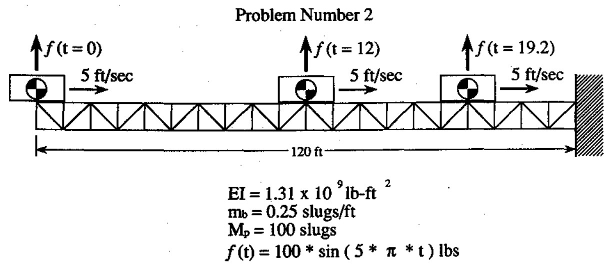


Fig. 3 Truss beam with moving pallet.

Equation (19) represents the incremental solution formula. Note that it has the form of a static equilibrium equation, with the large square matrix being the effective stiffness matrix. The external generalized force P acting on the beam is zero for this problem. Note also that the modal response and the pallet to beam interface force f are coupled together in the matrix equation. As was stated earlier, these variables are interrelated.

Since the pallet is translating, ϕ_b will change with each time step. Since the pallet will not be located on a truss structure model nodal point, except maybe for certain solution time steps, ϕ_b will be interpolated. The interpolation scheme is simplified by noting that the pallet is always on one finite element, thus, that element's displacement field is used. A simple beam bending finite element is shown in Fig. 4. The transverse displacement w at the location of the pallet mass is related to the beam element's nodal displacements through the following formula:

$$\begin{aligned} w &= (1/L^3)(L^3 - 3Lx^2 + 2x^3)q_1 \\ &\quad + (1/L^2)(L^2x - 2Lx^2 + x^3)q_2 \\ &\quad + (1/L^3)(3Lx^2 - 2x^3)q_3 - (1/L^2)(Lx^2 - x^3)q_4 \end{aligned} \tag{20}$$

Equation (20) is used to calculate ϕ_b by substituting ϕ_b for w and the appropriate eigenvectors for the nodal displacements q. This assures that the interpolation scheme is compatible with the assumptions of the finite element modeling. The transient solution can now be solved by use of Eq. (19). Note that in the calculation of the D and V terms of Eq. (19), the pallet to beam force from the previous time step is generalized, i.e., $\phi_{i,j}^T \, f_{j,n}$. For the solution of this problem, the eigenvector ϕ in this expression is the ϕ_b from the previous time step.

Computational Efficiency

This procedure does avoid the calculation of multiple eigensolutions; however, it requires the solution of a system of coupled equations that can typically be quite large. Also, a new solution (matrix inversion) is required each time step. Reference 6 used the approach of static condensation to greatly reduce the size of the required matrix inversions. As was pointed out in Ref. 6, if the changes in the effective stiffness matrix are localized, static condensation can greatly reduce the computational effort. This was the approach used in Refs. 2 and 3 and is also employed in this paper. As stated earlier, the form of Eqs. (14) and (19) is identical to the static equation of equilibrium:

$$[K]\{x\} = \{f\} \tag{21}$$

Equation (21) is partitioned into two sets of degrees of freedom:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{22}$$

Equation (22) now consists of two separate matrix equations. Static condensation uses algebra to solve for one solution subset by elimination of the other. In other words, static condensation is used to reduce out the first set of degrees of freedom x_1, resulting in the following equation for x_2:

$$[K_{22} - K_{21}K_{11}^{-1}K_{12}]\{x_2\} = \{f_2 - K_{21}K_{11}^{-1}f_1\} \tag{23}$$

This procedure is mathematically exact. The two solution subsets x_1 and x_2 are still dependent on each other. Note that this procedure required the inverse of partition K_{11}; however, by keeping the force degrees of freedom of Eq. (19) in the x_2 subset, this inversion is only needed once. That is, all modifications to the equations of motion will take place within the reduced system of equations given by Eq. (23). This was the method used by Refs. 2 and 3. After the solution of x_2, the solution to the first subset x_1 can be completed.

The solution technique, Eqs. (1–3), used in this paper simplifies the static condensation of Eq. (19). Note that performing the reduction on Eq. (19) with the modal displacements and velocities in the x_1 partition results in $K_{11} = I$. This condensation is, thus, a simple matrix multiply and add. The resulting equations are then reduced in a second condensation, this time reducing out the modal accelerations. Again, $K_{11} = I$. The reduced set of equations is given by Eq. (24):

$$\begin{bmatrix} I & & & \phi_p B_p \phi_p^T \\ & I & & -\phi_b B_b \phi_b^T \\ -I & I & I & \\ & & I & 0 \end{bmatrix} \begin{Bmatrix} w_p \\ w_b \\ \Delta w \\ f \end{Bmatrix} = \begin{Bmatrix} \phi_p D_p \\ \phi_b D_b \\ 0 \\ 0 \end{Bmatrix} \tag{24}$$

Equation (24) is again reduced down to only the interface force degree of freedom:

$$[\phi_p B_p \phi_p^T + \phi_b B_b \phi_b^T]\{f\} = \{\phi_p D_p - \phi_b D_b\} \tag{25}$$

The solution to Eq. (25) is a trivial task. The large system of equations has been reduced to a small fraction, actually only 1 DOF for example problem 2. The reduction did not even require a one-time large matrix inversion. The computational cost to modify and solve Eq. (25) at each time step is small. This solution results in the interface force acting between the truss structure and the pallet. Once this is known, the corresponding modal responses are solved for by use of Eqs. (7–9), i.e., the standard uncoupled linear solution.

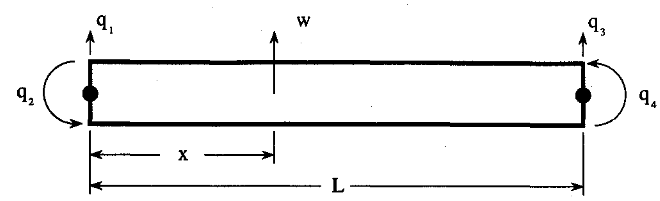


Fig. 4 Finite element beam.

Problem 2 Solution

The eigenvalues and frequencies for the Space Station beam, with zero pallet mass, are listed in Table 2. Since the pallet was to interface with all 20 of the beam degrees of freedom, residual modes were not used. It is not possible to truncate to fewer modes and then correct all degrees of freedom for truncated flexibility. Thus, in order to maintain total flexibility at all beam to pallet interface degrees of freedom, all 20 beam modes were retained.

The initial attempt to solve problem 2 with all beam modes proved to be more difficult than the authors had anticipated. Solving for the pallet to beam interface force appeared to be numerically unstable. The solution method of Ref. 4 neglected

Table 2 Problem 2 eigenvalues

Mode	Eigenvalue	Frequency, Hz
1	312.4	2.813
2	12,269.9	17.29
3	96,240.7	49.374
4	370,083.7	96.821
5	1,014,478.0	160.303
6	2,276,839.0	240.152
7	4,482,061.0	336.945
8	8,044,677.0	451.413
9	13,449,230.0	583.672
10	20,784,540.0	725.588
11	36,818,620.0	965.726
12	53,739,300.0	1166.719
13	78,970,310.0	1414.334
14	1.15035E + 08	1707.007
15	1.66106E + 08	2051.221
16	2.37603E + 08	2453.276
17	3.34688E + 08	2911.658
18	4.56162E + 08	3399.222
19	5.78470E + 08	3827.901
20	9.06014E + 08	4790.573

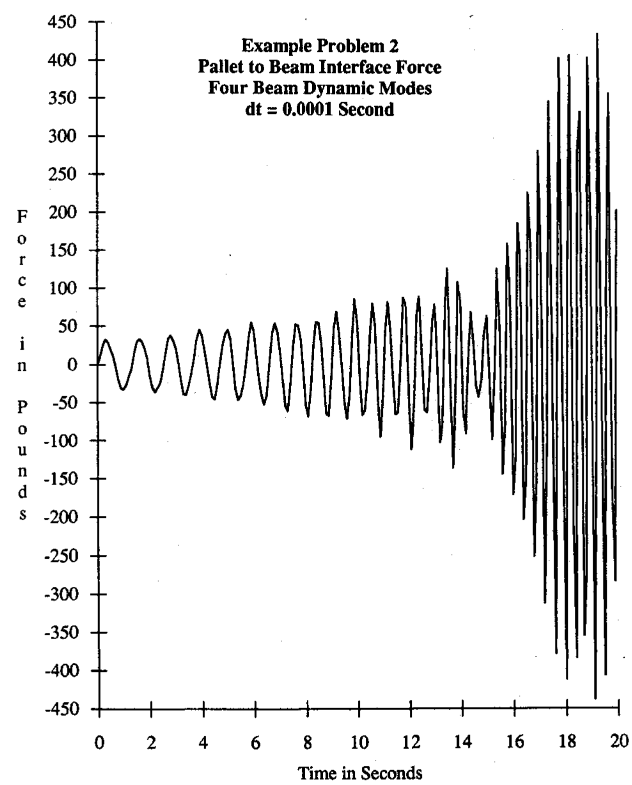


Fig. 5 Problem 2 solution.

various nonlinear terms that arise due to coupling between the beam transverse vibration and the translational motion of the mass. Since the previously mentioned solution method treats the pallet and beam as two separate modal systems, with the necessary constraints being applied between them, the method does account for such nonlinear effects. It is possible that these effects contribute to the numerical sensitivity. Another contribution may be the large range of eigenvalues being used (see Table 2).

Although this numerical sensitivity was not fully understood, the authors found that by treating the high-frequency beam modes quasistatically, as was suggested for residual modes, and using a rather small time step size, the solution was possible. Figure 5 shows the pallet to beam interface force solution for a time step size of 0.0001 s and treating only the first four beam modes as dynamic, i.e., modes 5–20, are treated quasistatically. The solution is plotted for every 0.1 s.

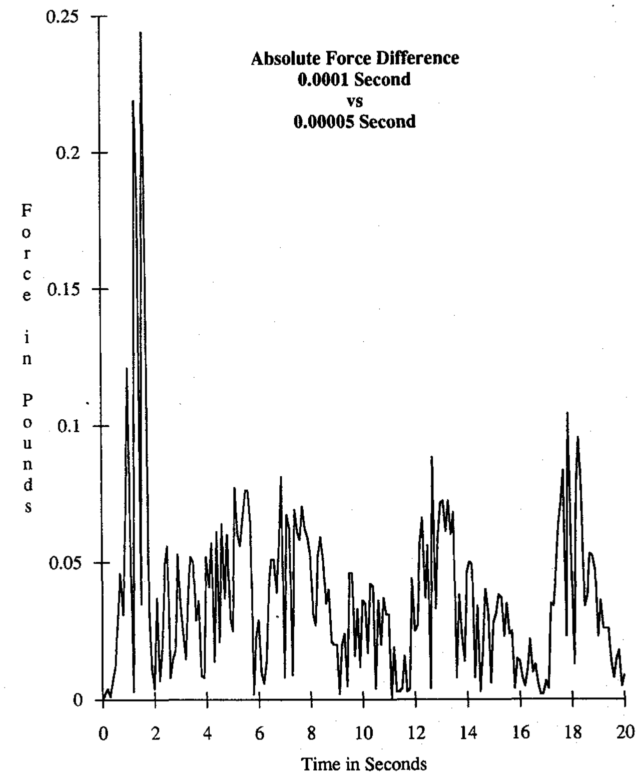


Fig. 6 Time step convergence test.

Using the 1.5 factor rule from Ref. 5, the system response should be accurate up to about 64.5 Hz. The time history mentioned earlier does not show any significant response frequency above 4 Hz. The solution convergence was tested both for dynamic modal content and time step size. Figure 6 plots the absolute force difference between time step sizes of 0.0001 and 0.00005 s. As can be seen, the maximum force difference is < ¼ lb. The modal content convergence was tested by comparing the force differences when keeping the first beam mode dynamic, the first four dynamic, and the first 10. Figures 7 and 8 illustrate this convergence using a time step of 0.0001 s. As can be seen, the problem is dominated by the first beam mode and is very well converged when only the first four beam modes are treated dynamically.

Validity of the Solution

The convergence to the solution of problem 2 was well treated and demonstrated. The derivation of the solution technique is rather rigorous and exact. However, the time history presented in Fig. 5 has to make physical sense before the reader will fully accept the method. A few items of review should help to accomplish this.

First, the frequency content of the resulting force time history must be reasonable. The fundamental frequency of the beam/pallet system is approximated easily by the following formula:

$$\frac{1}{\omega^2} = \frac{1}{\omega_b^2} + \frac{1}{\omega_p^2} \tag{26}$$

where ω^2 is the fundamental beam/pallet system eigenvalue, ω_b^2 is the first beam bending mode eigenvalue (i.e., the first mode of Table 2), and ω_p^2 is the eigenvalue resulting from the pallet mass on a massless beam (which is a function of time). Equation (26) results in frequencies of 0.73, 1.71, and 2.67 Hz for times 0.0, 12.0, and 19.2 s respectively. The frequency content taken from Fig. 5 is 0.81, 1.56, and 2.95 Hz at the respective times. The comparison is quite good and reinforces the fact that the problem is, for the most part, dominated by the first beam bending mode.

Second, the magnitude of the interface force after about 16 s may be deemed by some readers as suspicious. The force applied to the pallet has a frequency of 2.5 Hz. With Eq. (26), it can be shown that the beam/pallet system is tuned to 2.5 Hz at 17.76 s. It would not be surprising to see the interface force reach some higher magnitudes near this critical time. Figure 5 shows that this is exactly the case. After about 15 s, the peak force magnitude increases with each cycle, through about 18 s.

The expected force magnitude can be investigated further by a simplification of the problem. Assume the beam to be massless. The problem now reduces to a single-DOF oscillator on a time-varying spring. The equation of motion is written as follows:

$$M\ddot{w} + K(t)w = f(t) \tag{27}$$

Substituting in the specific variables for problem 2 results in the following:

$$100\ddot{w} + \frac{3*1.31*10^9}{(120-5t)^3}w = f(t) \tag{28}$$

This equation can be solved by any number of finite difference approaches. A modified Newmark-Beta method[1] is employed in this paper. The first solution to Eq. (28) involves a twang of the beam, i.e., the beam tip is displaced 0.25 in. and released with the pallet at the tip, moving 5 ft/s toward the cantilevered support. With no damping in the system and no applied force, $f(t) = 0$, the response is dependent on the initial strain energy in the beam. If the energy in the system was to remain constant, a force limit could be calculated for any pallet position or as a function of time. Figure 9 plots the solution to this single-DOF twang problem with the constant initial energy force limits superimposed.

As the pallet moves toward the stiffer portion of the beam, i.e., closer to the cantilevered support, the peak force magnitude increases. Also, it is apparent that energy is being added to the system, i.e., the response is exceeding the initial strain energy force limit. This additional energy must be coming from the mechanism that is increasing the single-DOF oscillator's spring stiffness. In the case of the moving pallet, this energy is generated from the various nonlinear forces that arise due to the coupling between the beam tranverse vibration and the translational motion of the mass.[4] If the multiple eigensolution techniques do in fact neglect such energy addition (or dissipation), their results could be far from accurate. What is worse, the relative magnitude of these errors (or even their very existence) may never be known to the analyst! The safest approach is to mimic the physics of the problem as closely as possible. The technique reported in this paper does this quite rigorously.

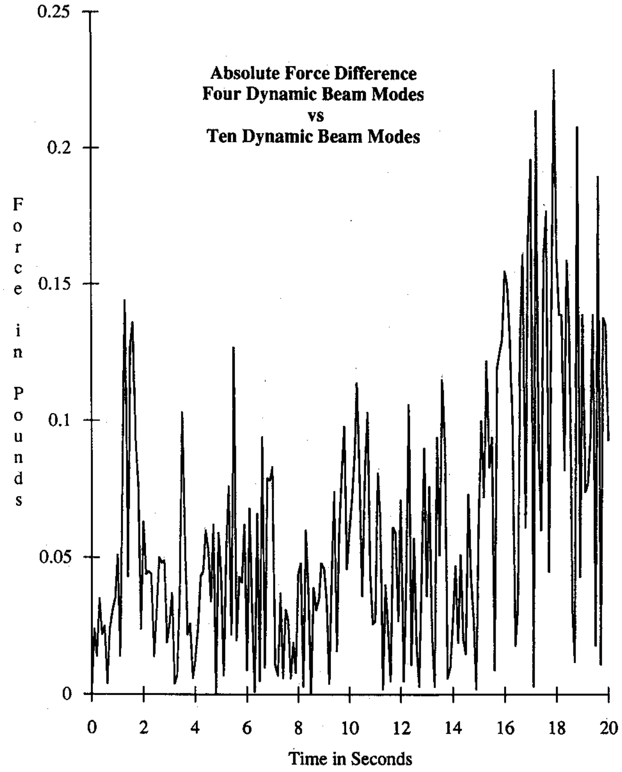


Fig. 8 Four modes vs 10.

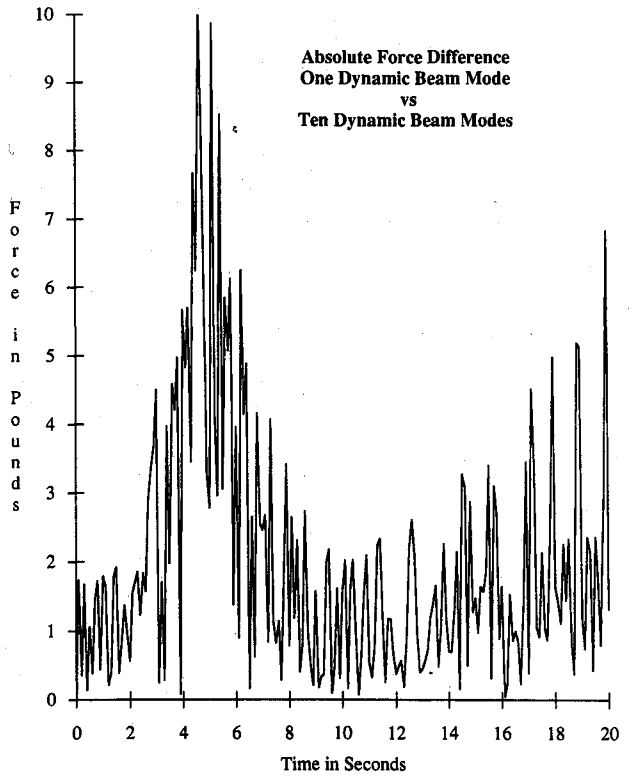


Fig. 7 One mode vs 10.

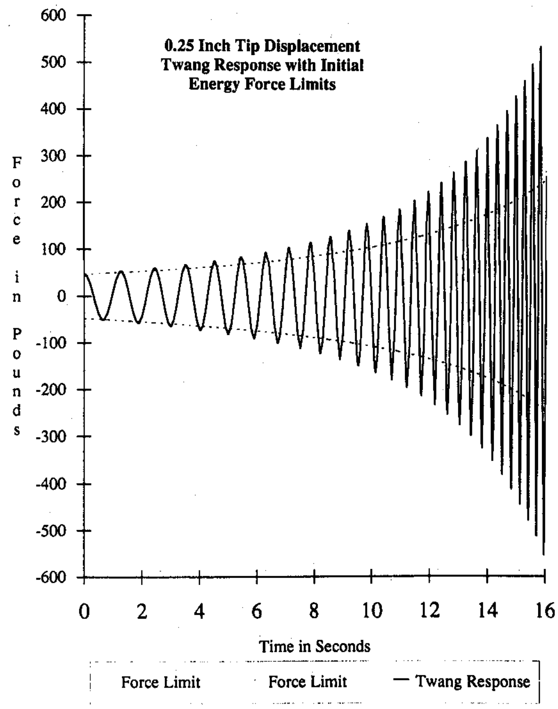


Fig. 9 Twang response with force limits.

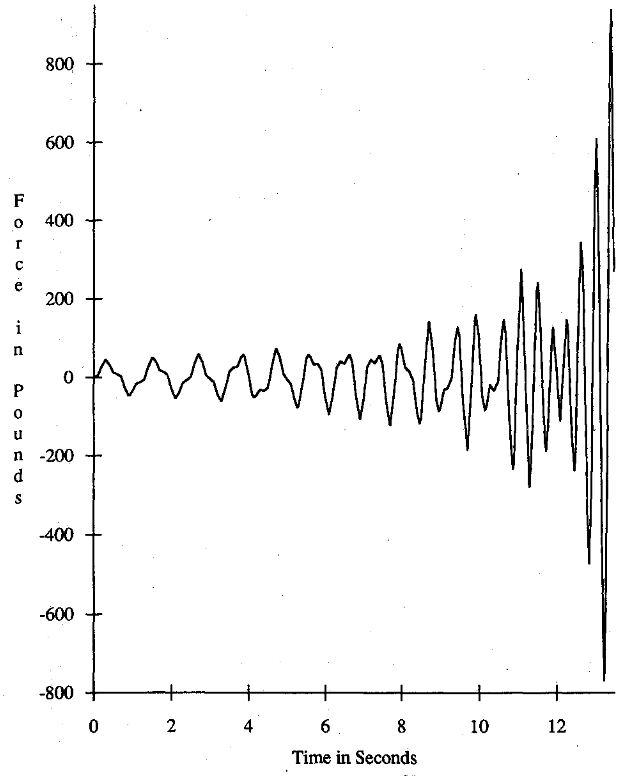


Fig. 10 Forced response-massless beam.

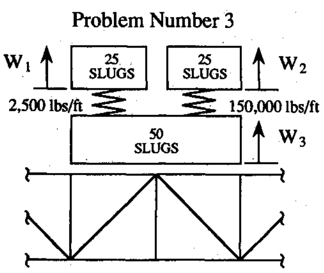


Fig. 11 Moving pallet with modal properties.

Figure 10 shows the solution to the massless beam problem except with the forcing function of problem 2. Again, as the pallet moves onto the stiffer portion of the beam, the pallet/beam interface force experiences large amplification. Based on the previous discussion, the results of problem 2, depicted in Fig. 5, are deemed to be very plausible.

Example Problem 3

The third example problem is an extension to problem 2. As illustrated in Fig. 11, the moving pallet is given modal properties that can dynamically interact with the truss structure. All other parameters are the same as in problem 2. The forcing function of problem 2, $f(t)$, acts on pallet DOF number 3. The truss structure and the pallet are treated as two independent modal systems, i.e., eigensolutions are found for each system. Note that for the truss structure this is the same solution used in problem 2 and also that the external generalized force P acting on the beam is zero.

Problem 3 Solution

The eigenvalues and frequencies for the Space Station moving pallet are listed in Table 3. Note that this pallet substructure has one rigid-body mode and two flex-body modes. The problem 3 time history solution for the beam to pallet interface force is plotted in Fig. 12. This solution includes the dynamics of the first four beam modes (all others being treated quasistatically) and the dynamics for all of the pallet

Table 3 Problem 3 pallet eigenvalues

Mode	Eigenvalue	Frequency, Hz
1	0.0000	0.0000
2	133.0832	1.8360
3	9016.9170	15.1130

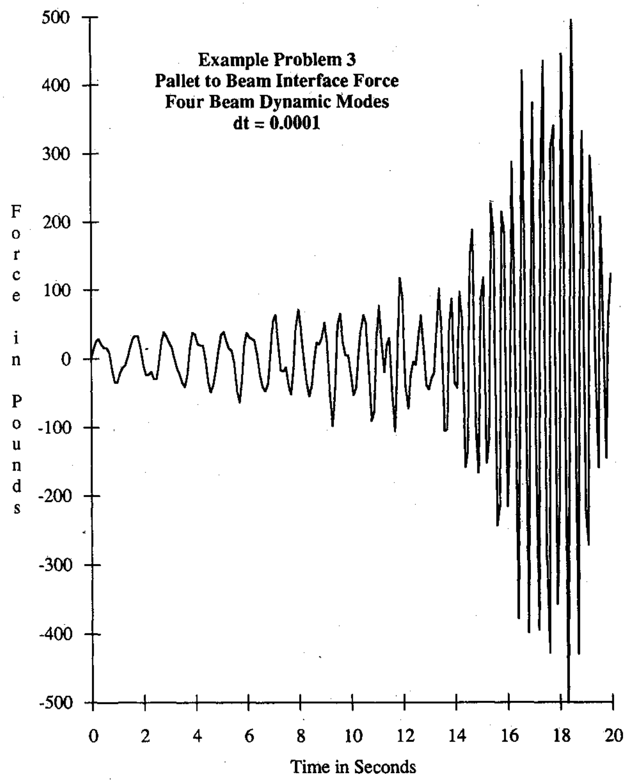


Fig. 12 Problem 3 solution.

modes. The solution is for an integration time step of 0.0001 s and is plotted for every 0.1 s. The influence of the pallet's dynamic response is apparent by comparing Fig. 12 to Fig. 5. This problem showed the same numeric sensitivity and convergence as problem 2.

Modal Truncation and Residual Modes

Use of residual modes to correct for modal truncation effects can be demonstrated by resolving problem 2. The solution's modal content is truncated down to the four lowest frequency modes, and the first eight beam degrees of freedom with which the pallet will interface with, i.e., nearest the tip, are corrected for their truncated flexibility and inertia. This results in eight residual modes, which are then treated quasistatically. The solution is executed with the converged time step of 0.0001 s. Figure 13 illustrates the differences between this solution and the solution with all beam degrees of freedom having their total flexibilities (i.e., the solution presented in Figure 5). As the pallet moves inboard on the beam, it will interface with all 20 beam degrees of freedom, the first eight of these having residual flex corrections. After 7.2 s, the pallet begins to interface with those degrees of freedom that suffer flexibility loss due to modal truncation. It would be expected that the solution would be accurate through 7.2 s and then would start to deviate.

Also plotted in Fig. 13 is the same solution with only the first four degrees of freedom corrected for truncated flexibility. In this solution, the pallet will interface with uncorrected degrees of freedom after only 2.4 s and is thus expected to deviate earlier than the previous solution. The results are as expected. This demonstration of use of residual modes would

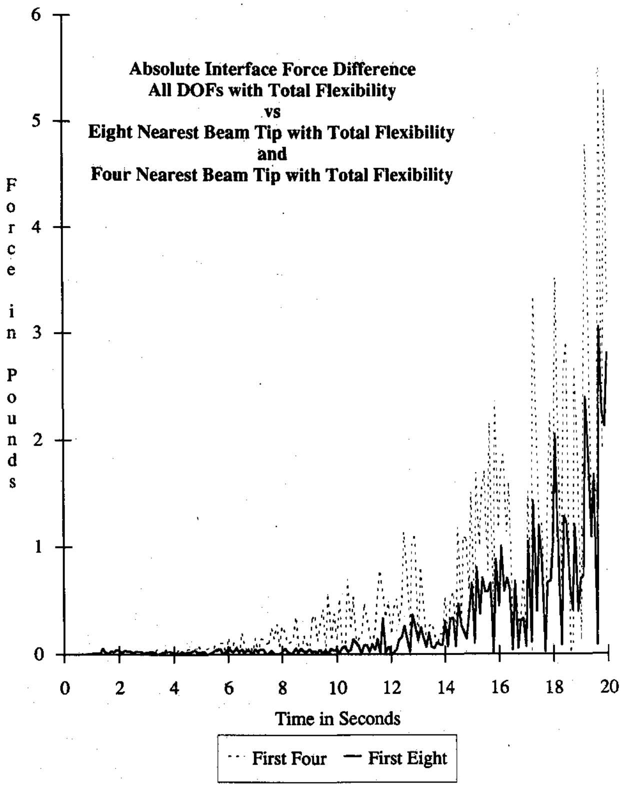


Fig. 13 Residual mode demonstration.

have been stronger if not for the fact that problem 2 is strongly dominated by the first beam mode. Thus, with this mode not being truncated, the flexibility corrections are somewhat small. However, the increasing error in the solution is still apparent after 7.2 s for eight residual modes and 2.4 s for four residual modes.

Summary

The solutions to the three example problems used the standard NASTRAN closed-form linear modal solution approach while using an invariant set of modal properties. The solutions presented in Ref. 4 employed multiple eigensolutions with interpolation schemes being used to calculate all of the required intermediate eigenvalues, eigenvectors, and eigenvector time derivatives. The methods of this paper avoid this complexity completely and should thus be less error prone. With acceptance that the modal synthesis methods of Ref. 5 are accurate, the only assumption in the solutions is that the forces due to time-varying structural parameters varied linearly between integration time steps. Solution convergence with time step size was demonstrated.

The particular treatment of the unknown force vector f due to time-variant structural properties resulted in a large set of coupled equations. Since these equations are updated each time step, repeated equation solutions are necessary. However, static condensation was used to greatly reduce computational effort, making this technique very efficient.

The principles of modal synthesis, presented in Ref. 5, were discussed. It was pointed out in this paper that the chosen time-variant modal properties can be formulated such that an accurate system modal representation, for any time in the solution, can be achieved.

Conclusions

This paper has presented a general solution methodology that can greatly simplify the solutions of a large class of complicated transient problems with time-varying properties. The technique was demonstrated with single-body and multiple-body example problems. The method of treating multiple bodies as separate modal substructures has proved to be very effective in the treatment of recontact loads between two spacecraft. The method's most obvious applications may be analyses involving docking and/or separation of various spacecraft structures. The method has found use in solving large structural transients with substructural interfaces that have deadbands.

The problem of the moving pallet on the Space Station beam had the simplification that the relative lateral pallet motion was constant. Therefore, this was a simple constraint on the problem, i.e., the drive mechanism maintained the motion, no matter what the energy requirements were. In a more detailed analysis, the capability and characteristics of drive mechanism would be modeled. In such an analysis, the location of the pallet could be part of the solution. Instead of an incremental time step solution, the resulting solution technique may require an iterative approach at each time step. The iterative solution would be executed not on the full set of modal equations, but on the rescued set that define all the necessary interface response. With this, the method could still be quite efficient. Iteration on each time step could also open the method to the treatment of other discrete nonlinearities, e.g., friction surfaces and various types of dampers.

In the same manner as the constraint between the pallet and the Space Station beam were treated, the method could also solve problems that involve rotating interface joints, i.e., articulated systems. Thus, the method could probe to be very powerful in the solution of kinematic systems where the flexible transient response of the structural elements is significant.

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